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# On the constrained elastic ring

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**Abstract.** The problem of a thin elastic ring contained within a smooth rigid cavity is considered for the case where the ring is subjected to a radial point load. The problem is approached as a moving intermediate boundary problem in the calculus of variations and a closed-form analytical solution is obtained. Numerical results are presented for several cases, revealing unstable behavior of the ring configuration.

## 1. Introduction

In many applications a protective lining is inserted into the interior of a cavity or cylindrical structure. As such linings are generally used for insulation or environmental purposes their behavior under load is of interest.

To date, several authors have considered various aspects of the problem of an elastic ring contained within a rigid cavity. These include the effects of circumferential load [1–4], inertial loading [5, 6], a contracting cavity wall [7], and external pressure [8, 9].

In the present work, the problem of an elastic ring contained within a cavity is considered for the case where the ring is subjected to a radially-directed point load. As the ring may be considered to be divided into two parts, one which is deflected away from the cavity wall and the other maintaining sliding contact, the problem is approached herein as a moving intermediate boundary-value problem in the calculus of variations with the "boundary" between the two "regions" of the ring being sought as part of the solution. The ring is modeled using the "arch equations" of [7], and the theorem of stationary potential energy is applied with the resulting transversality condition yielding the relation which defines the intermediate boundary. Numerical simulations are performed for several cases with the results presented as curves in the "load-deflection" and "load-boundary angle" spaces. Characteristic behavior of the ring configuration is discerned from the above results.

## 2. Problem formulation and analytical solution

Consider the circular elastic ring of thickness $\bar{h}$ contained within the cavity of radius $R$ shown in Fig. 1. The cavity wall shall be considered smooth and rigid while the ring, shown deflected by a radial point load of magnitude $\bar{Q}_0$, is divided into two regions. The first region, which shall be referred to as the "lift zone" or "lift region" is defined on the domain $\mathscr{D}_1$ corresponding to $0 \leqslant \theta \leqslant \phi$, where $\theta$ is the angular coordinate measured clockwise from the line of action of the applied load. The second region shall be referred to as the "contact zone" or "contact region" and corresponds to the portion of the ring in contact with the cavity wall.

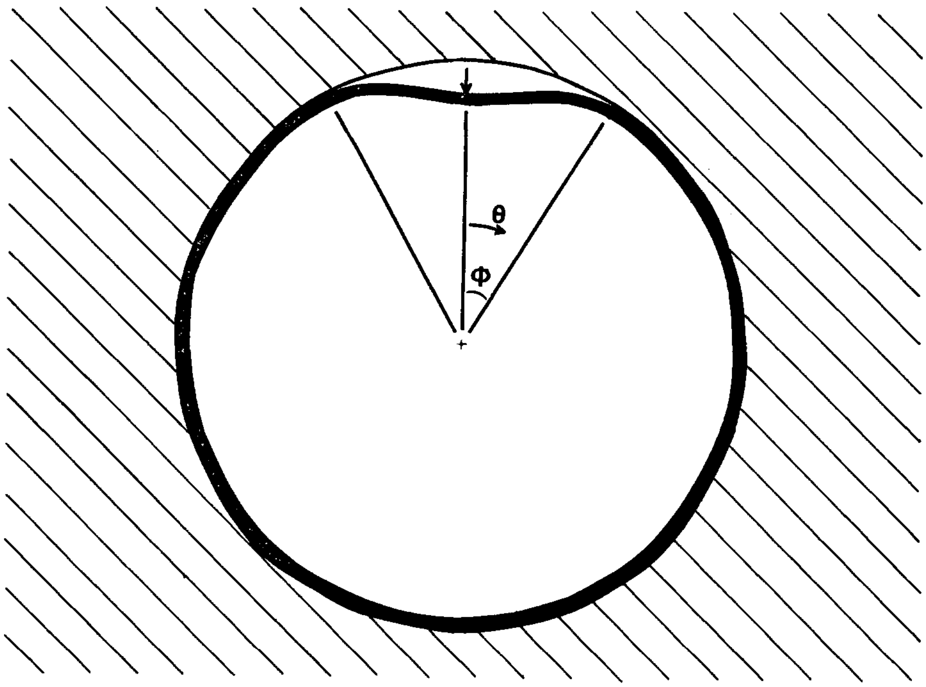


*Fig. 1.* Deflected elastic ring within rigid cavity.

It is defined on the domain $\mathscr{D}_2$ corresponding to $\phi \leqslant \theta \leqslant \pi$. We note that only half the ring need be analyzed as a result of the symmetry of the problem.

The problem shall be approached as a moving intermediate boundary-value problem in the calculus of variations, with the intermediate boundary, $\phi$, sought as part of the solution. The corresponding differential equations, associated boundary and interface conditions, and the transversality condition resulting from the moving intermediate boundary are found by incorporating a shallow-arch theory as the mathematical model and applying the theorem of stationary potential energy. In what follows, all displacements are normalized with respect to the radius of the cavity.

We shall begin by formulating an energy functional, Π, as follows:

$$\Pi = \sum_{i=1}^{2} [U_B^{(i)} + U_M^{(i)}] - \mathscr{W} + \Lambda, \tag{1}$$

where $U_B^{(i)}$ and $U_M^{(i)}$ correspond to the normalized bending and membrane energies, respectively, on $\mathscr{D}_i$, $\mathscr{W}$ corresponds to the work done by the normalized radial point load $Q_0 = \bar{Q}_0 R^2/D$, $D$ is the bending stiffness of the ring, and Λ is a "constraint functional" restricting the deflections of the portion of the ring in the "contact zone".

The explicit forms of these functionals are given by

$$U_B^{(1)} = \frac{1}{2}\int_0^{\phi} \kappa_1^2 \,\mathrm{d}\theta, \quad U_m^{(1)} = \frac{1}{2C}\int_0^{\phi} N_1^2 \,\mathrm{d}\theta, \tag{2a, b}$$

$$U_B^{(2)} = \frac{1}{2}\int_{\phi}^{\pi} \kappa_2^2 \,\mathrm{d}\theta, \quad U_m^{(2)} = \frac{1}{2C}\int_{\phi}^{\pi} N_2^2 \,\mathrm{d}\theta, \tag{2c, d}$$

$$\mathscr{W} = Q_0 w_1(0), \tag{2e}$$

and

$$\Lambda = \int_\phi^\pi \lambda w_2 \, d\theta, \tag{2f}$$

where using the arch equations of El-Bayoumy [7], the normalized curvature change of the portion of the ring on $\mathscr{D}_i$, $\kappa_i$ is given by

$$\kappa_i = w_i'' + w_i, \quad (i = 1, 2) \tag{3a}$$

and the normalized resultant membrane force on $\mathscr{D}_i$, $N_i$ is given by

$$N_i = -C[u_i' - w_i + \tfrac{1}{2} w_i'^2], \quad (i = 1, 2). \tag{3b}$$

In the above expressions $u_i$ (positive clockwise) and $w_i$ (positive inward) correspond to the normalized circumferential and radial deformations, respectively, of a material particle on the ring centerline. In addition, $C = 12/h^2$ is the normalized membrane stiffness of the ring where $h = \bar{h}/R \ll 1$ is the normalized thickness of the ring, $\lambda$ is a Lagrange multiplier, and $(\ )' \equiv d(\ )/d\theta$.

The theorem of stationary potential energy shall next be applied, which in the context of the present problem may be stated as

$$\delta \Pi = 0 \tag{4}$$

where $\delta$ corresponds to the variational operator. Substitution of (1), along with (2) and (3), into (4) yields the governing differential equations

$$\left.\begin{aligned} \kappa_i'' + \kappa_i + (N_i w_i')' + N_i &= p_i, \quad &(5)\\ N_i' &= 0, \quad &(6)\end{aligned}\right\} \theta \in \mathscr{D}_i \quad (i = 1, 2),$$

and

$$w_2 = 0, \quad \theta \in \mathscr{D}_2, \tag{7}$$

where

$$p_1 = 0 \quad \text{and} \quad p_2 = \lambda. \tag{8a, b}$$

We also obtain the boundary and matching conditions

$$[\kappa_1' + N_1 w_1']_{\theta=0} = Q_0, \tag{9a}$$

$$w_1'(0) = u_1(0) = 0, \tag{9b, c}$$

$$w_1(\phi) = w_2(\phi), \quad w_1'(\phi) = w_2'(\phi), \quad u_1(\phi) = u_2(\phi), \tag{10a, b, c}$$

$$N_1(\phi) = N_2(\phi), \tag{10d}$$

$$w_2(\pi) = 0, \quad w_2'(\pi) = u_2(\pi) = 0, \tag{11a, b, c}$$

and the transversality condition resulting from the vanishing of the coefficients of $\delta\phi$ in (4)

$$\Big[\tfrac{1}{2}(\kappa_1^2 - \kappa_2^2) + \kappa_1' w_1' - \kappa_2' w_2' - \kappa_1 w_1'' + \kappa_2 w_2'' - \lambda w_2$$

$$+ \frac{1}{2C}(N_1^2 - N_2^2) + N_1(u_1' + w_1'^2) - N_2(u_2' + w_2'^2)\Big]_{\theta=\phi} = 0. \tag{12}$$

The quantity $p_2$ may be identified as the contact pressure $p$, thus

$$p_2 = p = \lambda, \quad \theta \in \mathscr{D}_2. \tag{13}$$

Upon integrating (6) and imposing (10d) we find that

$$N_1 = N_2 = N_0 = \text{constant}. \tag{14}$$

If we now take (13) and (14) into account the governing differential equations simplify to

$$\mathscr{L}\{w_i\} = p\hat{\delta}_{i2} - N_0, \quad \theta \in \mathscr{D}_i \quad (i = 1, 2), \tag{15a, b}$$

and

$$w_2 = 0, \tag{15c}$$

where

$$\mathscr{L} = \frac{d^4}{d\theta^4} + (N_0 + 2)\frac{d^2}{d\theta^2} + 1, \tag{16}$$

and $\hat{\delta}_{ij}$ corresponds to Kronecker's delta. The associated boundary and matching conditions similarly reduce to

$$w_1'''(0) = Q_0, \quad w_1'(0) = 0, \tag{17a, b}$$

$$w_1(\phi) = w_1'(\phi) = 0, \tag{18a, b}$$

with (7), (9c), (10c), and (11c) combined to eliminate $u_1$ and $u_2$ and form the following equivalent expression in terms of $w_1$:

$$\int_0^\phi (w_1 - \tfrac{1}{2}w_1'^2)\,d\theta = \pi N_0/C. \tag{19}$$

Finally, the transversality condition (12) reduces to the simple form

$$w_1''(\phi) = 0. \tag{20}$$

The condition (20), when combined with (18a), is seen to specify that the bending moment vanishes at the "lift zone"–"contact zone" interface. The system (15) together with conditions (17)–(20) constitute a moving intermediate boundary problem for the deflection $w_1(\theta)$, the membrane force $N_0$, and the interface angle $\phi$. The corresponding solution is given by

$$w_1(\theta) = G_0(\alpha, \phi, Q_0) \cos \alpha\theta - F_0(\alpha, \phi, Q_0) \cos \theta/\alpha - \frac{Q_0 \alpha}{\alpha^4 - 1}(\sin \alpha\theta - \alpha^2 \sin \theta/\alpha) - N_0, \quad \theta \in \mathscr{D}_1 \tag{21}$$

$$w_2(\theta) = 0, \quad \text{and} \quad p(\theta) = N_0, \quad \theta \in \mathscr{D}_2 \tag{22a, b}$$

where

$$\alpha^2 = \tfrac{1}{2}[N_0 + 2 + \sqrt{N_0(N_0 + 4)}] > 1 \quad \text{or} \quad N_0 = (\alpha^2 - 1)^2/\alpha^2, \tag{23a, b}$$

$$G_0(\alpha, \phi, Q_0) = \frac{Q_0 \alpha}{(\alpha^4 - 1) \sin \alpha\phi}\left[\frac{F_1(\alpha, \phi)}{G_1(\alpha, \phi)} \sin \phi/\alpha - \cos \alpha\phi + \cos \phi/\alpha\right] + \frac{N_0 \sin \phi/\alpha}{G_1(\alpha, \phi)}, \tag{24a}$$

$$F_0(\alpha, \phi, Q_0) = \frac{\alpha^2}{G_1(\alpha, \phi)}\left[Q_0 \frac{\alpha}{\alpha^4 - 1} F_1(\alpha, \phi) + N_0 \sin \alpha\phi\right], \tag{24b}$$

$$F_1(\alpha, \phi) = 1 - \alpha^2 \sin \alpha\phi \sin \phi/\alpha - \cos \alpha\phi \cos \phi/\alpha, \tag{24c}$$

and

$$G_1(\alpha, \phi) = \cos \alpha\phi \sin \phi/\alpha - \alpha^2 \sin \alpha\phi \cos \phi/\alpha. \tag{24d}$$

Upon substitution of (21), the conditions (19) and (20) may be written as explicit functions of $\alpha$, $\phi$, and $Q_0$. We thus have

$$G_0(\alpha, \phi, Q_0)\frac{\sin \alpha\phi}{\alpha} - F_0(\alpha, \phi, Q_0)\alpha \sin \phi/\alpha + Q_0\left[\frac{\cos \alpha\phi - \alpha^4 \cos \phi/\alpha}{\alpha^4 - 1} + 1\right] - \tfrac{1}{2}Z_0(\alpha, \phi, Q_0) = N_0(\phi + \pi/C) \tag{25}$$

and

$$Q_0 H_1(\alpha, \phi) = N_0 H_2(\alpha, \phi) \tag{26}$$

where

$$Z_0(\alpha, \phi, Q_0) = \frac{\alpha G_0^2}{4}[\beta_1 - \sin\beta_1] - F_0 G_0 \phi \left[\frac{\sin\psi_1}{\psi_1} - \frac{\sin\psi_2}{\psi_2}\right] + \frac{F_0^2}{4\alpha}(\beta_2 - \sin\beta_2)$$

$$+ Q_0^2 \frac{\alpha^4 \phi}{(\alpha^4 - 1)^2}\left[1 + \frac{\sin\beta_1}{2\beta_1} + \frac{\sin\beta_2}{2\beta_2} - \frac{\sin\psi_1}{\psi_1} - \frac{\sin\psi_2}{\psi_2}\right]$$

$$+ Q_0 G_0 \frac{\alpha^3 \phi}{\alpha^4 - 1}\left[\frac{\sin^2 \alpha\phi}{\alpha\phi} + \frac{\cos\psi_2}{\psi_2} + \frac{\cos\omega_1}{\psi_1} - \frac{2\alpha^3}{(\alpha^4 - 1)\phi}\right]$$

$$+ Q_0 F_0 \frac{\alpha\phi}{\alpha^4 - 1}\left[\frac{\sin^2 \phi/\alpha}{\phi/\alpha} + \frac{\cos\psi_2}{\psi_2} - \frac{\cos\psi_1}{\psi_1} + \frac{2\alpha}{(\alpha^4 - 1)\phi}\right], \quad (27)$$

$$H_1(\alpha, \phi) = \alpha(\cos\alpha\phi - \cos\phi/\alpha), \quad (28a)$$

$$H_2(\alpha, \phi) = \alpha^2 \cos\alpha\phi \sin\phi/\alpha - \sin\alpha\phi \cos\phi/\alpha, \quad (28b)$$

$$\beta_1 = 2\alpha\phi, \quad \beta_2 = 2\phi/\alpha, \quad (29a, b)$$

$$\psi_1 = \frac{\alpha^2 - 1}{\alpha}\phi, \quad \psi_2 = \frac{\alpha^2 + 1}{\alpha}\phi, \quad (29c, d)$$

and $G_0$, $F_0$, $F_1$, and $G_1$ are given by (24a–d) respectively. Equations (25) and (26) constitute a pair of coupled nonlinear algebraic equations in $\alpha$, $\phi$, and $Q_0$ which may be solved simultaneously to yield the load-"interface" angle and load-deflection response for a given structure.

## 3. Numerical results

As stated at the end of the previous section, equations (25) and (26) constitute a coupled pair of nonlinear algebraic equations, the roots of which yield the locus of points which define the "equilibrium paths" in the "load-interface angle space" for the corresponding structure. The associated "crown point" deflections $\Delta_0 \equiv w(0)$ can then be obtained using equation (21), thus generating the image of the "equilibrium path" in the corresponding "load-deflection space". This is done by first substituting equation (26) into equation (25) which eliminates $Q_0$ and results in a single nonlinear algebraic equation in $\alpha$ and $\phi$. The values of $\alpha$ corresponding to desired values of $\phi$ are then found using the bisection technique, with the value of $Q_0$ associated with each $(\alpha, \phi)$ pair then obtained from (26). Finally, the corresponding value of $\Delta_0$ is evaluated using (21). Results are displayed in Figs. 2 and 3, for several values of the normalized stiffness $C$.

The first feature that may be observed from these figures is the existence of a maximum or "critical" load $Q_c$ at which point "snap-thru" takes place with the crown-point deflection

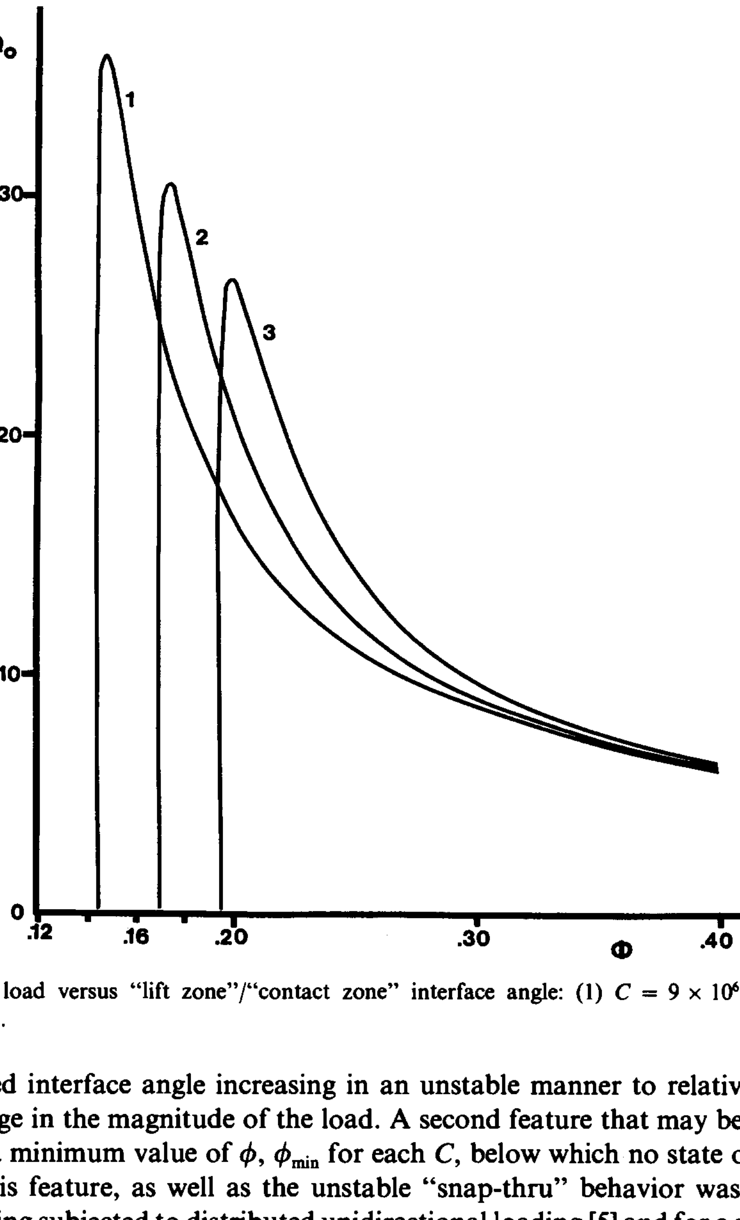


*Fig. 2.* Applied load versus "lift zone"/"contact zone" interface angle: (1) $C = 9 \times 10^6$, (2) $C = 4 \times 10^6$, (3) $C = 2 \times 10^6$.

and associated interface angle increasing in an unstable manner to relatively large values with no change in the magnitude of the load. A second feature that may be observed is the existence of a minimum value of $\phi$, $\phi_{\min}$ for each $C$, below which no state of equilibrium is observed. This feature, as well as the unstable "snap-thru" behavior was observed for a constrained ring subjected to distributed unidirectional loading [5] and for a parallel problem of a ring with initial clearance from the cavity wall [6]. Unlike for these cases, however, the minimum $\phi$ corresponding to the present problem is reached before $Q_0$ reaches $Q_c$ as the load is increased.

The implications of the presence of $\phi_{\min}$, and of the general shapes of the curves presented, offer the following interpretation. Consider the "equilibrium paths" corresponding to the ring where $C = 2 \times 10^6$ as an example. As we proceed up the $Q_0 - \phi$ path we note that the interface angle initially decreases slightly with increasing $Q_0$ until $\phi \approx 0.193$ corresponding to a "closing" or shrinking of the "lift zone" [0, $\phi$]. This would seem to imply that at low values of the applied load, and corresponding crown-point deflection, the ring initially "shrinks" with very little flexure. As the load is subsequently increased, the crown-point

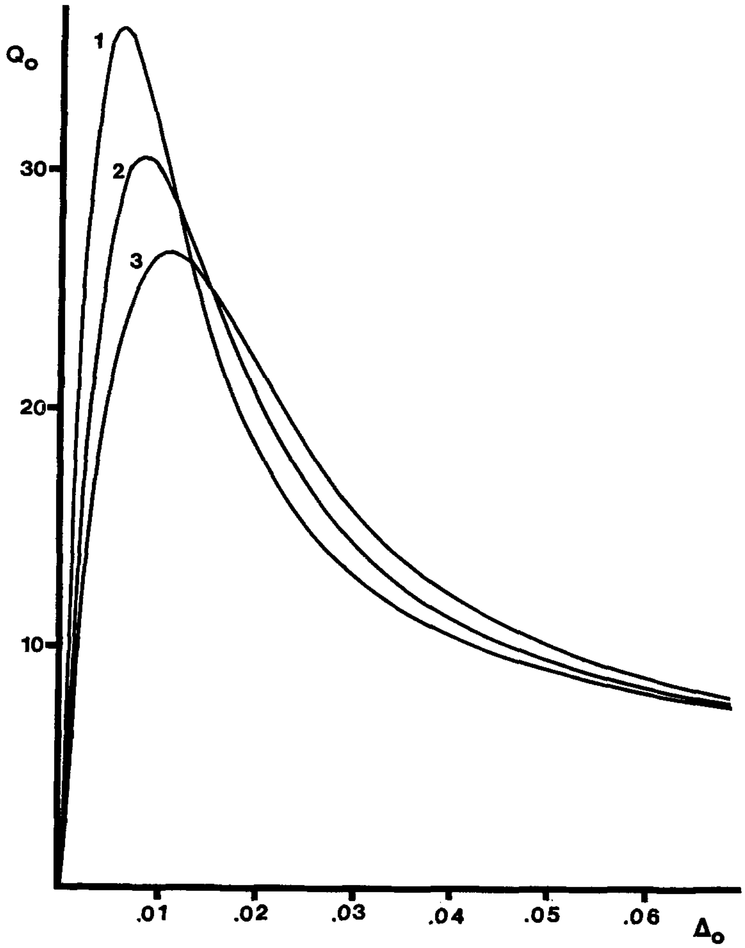


*Fig. 3.* Applied load versus "crown point" deflection: (1) $C = 9 \times 10^6$, (2) $C = 4 \times 10^6$, (3) $C = 2 \times 10^6$.

deflection increases monotonically with the increased flexure of the ring causing the "contact zone" to grow, thus reducing the size of the "lift zone" (i.e., $-\phi$ decreases) until the minimum value of $\phi$ is reached. At this point the ring begins to lift away from the wall of the cavity with the crown-point deflection still increasing, the process occurring in a stable manner until $Q_0 = Q_c \approx 26.5$ ($\phi \approx 0.0198, \Delta_0 \approx 0.0108$). At this point "snap-thru" occurs with the ring lifting away from the cavity wall in an unstable fashion.

Though not shown, we note that the normalized resultant membrane force, $N_0$, reaches a maximum value of $N_0 \approx 338.$ ($\Delta_0 \approx 0.0175$), $N_0 \approx 447.$ ($\Delta_0 \approx 0.0131$), and $N_0 \approx 618.$ ($\Delta_0 \approx 0.00955$), for $C = 2 \times 10^6$, $4 \times 10^6$, and $9 \times 10^6$ respectively.

Upon comparing the equilibrium paths corresponding to the different values of $C$ considered, we may note that the larger the stiffness of the ring the smaller is the minimum interface angle and, in general, the more "leftward" the entire curve. We may also note that as $C$ increases, the critical load increases and the "peak" of the corresponding curve becomes sharper. We further note that the "lifting" of the ring during "snap-thru" appears to be relatively more severe for rings of greater stiffness as a consequence of the larger amount of strain energy released during the process.

## 4. Concluding remarks

The problem of a thin elastic ring contained within a smooth rigid cavity and subjected to a radial point load has been addressed as a moving intermediate boundary problem in the calculus of variations. A closed-form analytical solution was obtained and the results of numerical studies employing that solution presented for several cases. The results, presented in the form of load-interface angle and load-deflection plots, revealed unstable "snap-thru" behavior of the ring for critical values of the applied load, as well as the existence of a minimum interface angle implying other qualitative features of the ring behavior under this type of loading and constraint.

## References


1. P.T. Hsu, J. Elkon and T.H.H. Pian, Note on the instability of circular rings confined to a rigid boundary, *ASME J. Appl. Mechs.* (Sept. 1964) 559–562.
2. S.J. McMinn and H.C. Chan, The stability of a uniformly compressed ring surrounded by a rigid circular surface, *Int. J. Mech. Sci.* 8 (1966) 433–442.
3. L.L. Bucciarelli Jr. and T.H.H. Pian, Effect of initial imperfections on the instability of a ring confined to an imperfect rigid boundary, *ASME J. Appl. Mechs.* (Dec. 1967) 979–984.
4. R. Chicurel, Shrink buckling of thin circular rings, *ASME J. Appl. Mechs.* (Sept. 1968) 608–610.
5. G. Herrmann and E.A. Zagustin, Stability of an elastic ring in a rigid cavity, *ASME J. Appl. Mechs.* (June 1967) 263–270.
6. R.D. McGhie and D.O. Brush, Deformation of an inertia-loaded thin ring in a rigid cavity with initial clearance, *Int. J. Solids and Structures* 7 (1971) 1539–1553.
7. L. El-Bayoumy, Buckling of a circular elastic ring confined to a uniformly contracting boundary, *ASME J. Appl. Mechs.* (1972) 758–766.
8. T.J. Lardner, On the nonbuckling of a circular ring under a "wrapping" loading, *ASME J. Appl. Mechs.* (Dec. 1980) 973–974.
9. S. Kyriakides and S.K. Youn, On the collapse of circular confined rings under external pressure, *Int. J. Solids and Structures* (1984) 699–713.